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Monopole-based quantization: a programme

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Abstract

We describe a programme to quantize a particle in the field of a (three dimensional) magnetic monopole using a Weyl system. We propose using the mapping of position and momenta as operators on a quaternionic Hilbert module following the work of Emch and Jadczyk.

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1 Introduction: several birds with a stone

Quantum kinematics in the field of a magnetic monopole allows for angular momentum-isospin coupling, in particular spin-isospin and orbit-isospin couplings. However, to our knowledge the rich conceptual and mathematical structures associated to angular momentum-isospin coupling have gone unnoticed so far in deformation quantization theory.

The aim of this contribution is to set out the basis for a rigorous investigation of those structures in canonical quantization, up to defining the pertinent monopole-based Weyl systems and star products. We note that angular momentum-isospin coupling is a feature of high physical interest; for a current example of practical application, we refer to graphene edge model [1]. Rather than presenting novel results we will outline, with the detail permitted by the total length limit, a general framework in which it will be possible to use the monopole for a variety of investigations both from the physical and mathematical point of view.

A classical particle in the field of a magnetic monopole of unitary charge is described, in proper units, by the Poisson structure:

$$\begin{aligned} \{p_i, x^j\} &= \delta_i^j \\ \{x^i, x^j\} &= 0 \\ \{p_i, p_j\} &= -\frac{1}{2}\epsilon_{ijk}\frac{x^k}{\|\mathbf{x}\|^2} \end{aligned} \tag{1.1}$$

This Poisson structure is position dependent and therefore its quantization is not trivial, but extremely rich! The right mathematical framework for the quantization of this structure is the formulation of quantum mechanics on a quaternionic Hilbert space given by Emch and Jadczyk over 10 years ago [2]. Since this does not seem to be common knowledge, at some point we summarize the findings of that paper, insofar they are useful for our purposes.

Prior to doing the above, we exhibit another problem of principle, in some sense dual to quantization in the field of a monopole, whose resolution hangs from the same mathematical thread. Thus the article is organized as follows. An *ab initio* calculation, using the Kirillov coadjoint picture [3] allows to regard the photon as a *classical* elementary physical system for the inhomogeneous Lorentz group $\mathcal{P}$. On the arena of phase space, this turns out to be formally dual (exchanging position and momenta) to the orbit-isospin coupling system. Section 3 deals with the structure of the gCCR (generalized canonical commutation relations) on the quaternionic Hilbert space. In Section 4, we show how the Emch–Jadczyk (EJ) calculus provides us the necessary tools for quantization of the above indicated systems. Quantization and dequantization proper are sketched in the next section. In Section 6 we give the conclusion and outlook for construction of the star product describing the scalar particle-monopole system.

Table 1: The coadjoint action $\mathrm{Coad}\,(\exp X)y$

X	$-a^0 H$	$\mathbf{a}\cdot\mathbf{P}$	$\alpha\mathbf{m}\cdot\mathbf{J}$	$\zeta\mathbf{n}\cdot\mathbf{K}$
h	h	h	h	$(\cosh\zeta)h-(\sinh\zeta)\mathbf{n}\cdot\mathbf{p}$
$\mathbf{p}$	$\mathbf{p}$	$\mathbf{p}$	$R_{\alpha\mathbf{m}}\,\mathbf{p}$	$\mathbf{p}-(\sinh\zeta)h\mathbf{n}+(\cosh\zeta-1)(\mathbf{n}\cdot\mathbf{p})\mathbf{n}$
$\mathbf{j}$	$\mathbf{j}$	$\mathbf{j}+\mathbf{a}\times\mathbf{p}$	$R_{\alpha\mathbf{m}}\,\mathbf{j}$	$(\cosh\zeta)\mathbf{j}+(\sinh\zeta)\mathbf{n}\times\mathbf{k}-(\cosh\zeta-1)(\mathbf{n}\cdot\mathbf{j})\mathbf{n}$
$\mathbf{k}$	$\mathbf{k}-a^0\mathbf{p}$	$\mathbf{k}+h\mathbf{a}$	$R_{\alpha\mathbf{m}}\,\mathbf{k}$	$(\cosh\zeta)\mathbf{k}-(\sinh\zeta)\mathbf{n}\times\mathbf{j}-(\cosh\zeta-1)(\mathbf{n}\cdot\mathbf{k})\mathbf{n}$

2 The orbit method for photons

The unique Poincaré-invariant Stratonovich–Weyl (de)quantizer and Moyal product on the phase space $T^*\mathbb{R}^3\times\mathbb{S}^2\simeq\mathbb{R}^6\times\mathbb{S}^2$ for massive relativistic particles with spin (degenerating to $T^*\mathbb{R}^3$ for spinless particles) was constructed in [4] with help of results in [5]. Its practical interest is nil since (contrary to the Galilean case) it breaks down as soon as one introduces interaction. However, this construction was an important matter of principle: the formalism based on this Moyal product is equivalent to relativistic quantum mechanics (and of course participates in its flaws). In particular, it gave geometric quantization on phase space and relativistic Wigner functions, establishing the bridge between the Kirillov coadjoint orbit picture and the Wigner theory of unitary irreps for the Poincaré group $\mathcal{P}$. For massless particles, although we knew the arrival point as well (see [6] for a modern, streamlined treatment), we were stumped. The time has come to revisit the question.

We describe coadjoint for the splitting group $\widetilde{\mathcal{P}}_0$ of the Poincaré group; this is just the universal covering $T_4\ltimes SL(2,\mathbb{C})$, without nontrivial extensions [7]. The Lie algebra of $\widetilde{\mathcal{P}}_0$ is generated by ten elements H,P^i,J^i,K^i (for $i=1,2,3$) corresponding respectively to time translations, space translations, rotations and pure boosts. Write elements of $\widetilde{\mathcal{P}}_0$ in the standard form

$$g=\exp(-a^0H+\mathbf{a}\cdot\mathbf{P})\,\exp(\zeta\,\mathbf{n}\cdot\mathbf{K})\,\exp(\alpha\,\mathbf{m}\cdot\mathbf{J}),$$

where $a=(a^0,\mathbf{a})\in T_4$, $\mathbf{n}$ and $\mathbf{m}$ are unit 3-vectors, $\zeta\geq 0$ and $0\leq\alpha\leq 2\pi$, with the understanding that $\exp(2\pi\mathbf{m}\cdot\mathbf{J})=-1_2$ in $SL(2,\mathbb{C})$ for all $\mathbf{m}$. The coadjoint action of $\widetilde{\mathcal{P}}_0$ on $\mathfrak{p}^*$ can be derived from the well-known commutation relations for the generators. Let h be the linear coordinate on $\mathfrak{p}^*$ associated to $H\in\mathfrak{p}$, and similarly let p^i,j^i,k^i be the coordinates associated to P^i,J^i,K^i ($i=1,2,3$). The action in these coordinates is given in Table 1.

The orbits of this action arise as level sets of two *Casimir functions* C_1,C_2 on $\mathfrak{p}^*$, which are easy to obtain explicitly (or to guess from other treatments). Let $p=(h,\mathbf{p})$ be the *energy-momentum* 4-vector and $w=(w^0,\mathbf{w})$ the *Pauli–Lubański* 4-vector given by

$$w^0=\mathbf{j}\cdot\mathbf{p},\qquad \mathbf{w}=\mathbf{p}\times\mathbf{k}+h\mathbf{j}.$$

From Table 1 one verifies that w^0 transforms like h and $\mathbf{w}$ like $\mathbf{p}$ under the coadjoint

action; in particular, under Coad $\big(\exp(\zeta\mathbf{n}\cdot\mathbb{K})\big)$:

$$\begin{aligned} w^0 &\mapsto (\cosh\zeta)w^0 - (\sinh\zeta)\mathbf{n}\cdot\mathbf{w}, \\ \mathbf{w} &\mapsto \mathbf{w} - (\sinh\zeta)w^0\mathbf{n} + (\cosh\zeta - 1)(\mathbf{n}\cdot\mathbf{w})\mathbf{n}. \end{aligned}$$

Thus the Casimir functions we seek are

$$C_1 := (pp) = -h^2 + \mathbf{p}\cdot\mathbf{p}, \qquad C_2 := (ww) = -(\mathbf{j}\cdot\mathbf{p})^2 + \|\mathbf{p}\times\mathbf{k} + h\,\mathbf{j}\|^2.$$

Notice that p and w are orthogonal in the Minkowski sense: $(pw) = 0$. Let us focus on the shape of light-like coadjoint orbits, for which $C_1 = 0$. For physical reasons (no 'continuous-spin' representations) we take the momentous decision of stipulating that w is parallel to p. Clearly $\mathbf{p}\in\mathbb{R}^3\setminus\{0\}$ (the origin is an orbit). We can postulate $\mathbf{q} := \mathbf{k}/h$, which is well defined, and takes all values in $\mathbb{R}^3$, and everything is determined:

$$h = \|\mathbf{p}\|, \qquad \mathbf{p} = \mathbf{p}, \qquad \mathbf{j} = \lambda\,\frac{\mathbf{p}}{\|\mathbf{p}\|} + \mathbf{q}\times\mathbf{p}, \qquad \mathbf{k} = \|\mathbf{p}\|\,\mathbf{q},$$

where $\lambda\mathbf{p}/\|\mathbf{p}\|$ plays the role of the spin, with the helicity $\lambda = \mathbf{j}\cdot\mathbf{p}/\|\mathbf{p}\|$ being the projection of the total angular momentum $\mathbf{j}$ on the momentum. Therefore the orbit is 6-dimensional, and isomorphic to $\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\simeq\mathbb{R}^3\times\mathbb{R}_+\times\mathbb{S}^2$. This non-trivial topology has non-trivial consequences.

By the general theory, the Poisson bracket is given by

$$\{f,g\} = c^k_{ij}\,\frac{\partial f}{\partial x_i}\,\frac{\partial g}{\partial x_j}\,x_k.$$

The commutation relations for the generators yield:

$$\{p^i, q^j\} = \{p^i, h^{-1}k^j\} = h^{-1}\{p^i, k^j\} = -\delta_{ij}. \tag{2.1}$$

On the other hand,

$$\begin{aligned} \{q^i, q^j\} &= \{h^{-1}k^i, h^{-1}k^j\} = h^{-2}\{k^i, k^j\} + h^{-1}k^j\{k^i, h^{-1}\} + h^{-1}k^i\{h^{-1}, k_j\} \\ &= h^{-2}(-\epsilon^{ij}_k j^k - q^j p^i + q^i p^j) = -\lambda\epsilon^{ij}_k\,\frac{p^k}{\|\mathbf{p}\|^3}; \end{aligned} \tag{2.2}$$

which is dual to the Poisson structure (1.1) upon the exchange $p\leftrightarrow q$.

The coordinates in (2.2) are *not* canonical coordinates (Darboux coordinates do not exist globally, but $d^3\mathbf{q}\,d^3\mathbf{p}$ is a global Liouville measure). All this agrees nicely with the analysis in [8]. We can recover from Table 1 the expression of the coadjoint action of $\widetilde{\mathcal{P}}_0$ on the orbit in terms of the coordinates $(\mathbf{p},\mathbf{q})$. There is no need to rewrite the action on $\mathbf{p}$. Also, we readily obtain:

$$\begin{aligned} \exp(-a^0 H)\triangleright\mathbf{q} &= \mathbf{q} - \frac{a_0\mathbf{p}}{\|\mathbf{p}\|} \\ \exp(\mathbf{a}\cdot\mathbf{P})\triangleright\mathbf{q} &= \mathbf{q} + \mathbf{a} \\ \exp(\alpha\mathbf{m}\cdot\mathbf{J})\triangleright\mathbf{q} &= R_{\alpha\mathbf{m}}\mathbf{q}. \end{aligned}$$

These formulae conform to our intuition as to how a relativistic particle should behave. They seem to indicate that, provided one allows for non-commutativity, the 'photon' (a massless relativistic particle in general) is in some sense a localizable particle, since the full Euclidean group is realized on a set of coordinate variables. (*Pace* the founding fathers: in the old paper [9] Wightman writes that no such a realization can exist; but he assumes commuting coordinates.) The symplectic form corresponding to (2.1) and (2.2) is given by:

$$\omega = dq^i \wedge dp^i - \lambda \epsilon_{ijk} \frac{p^k \, dp^i \wedge dp^j}{\|\mathbf{p}\|^3}.$$

This is exactly the one of the magnetic monopole, with the roles of $\mathbf{q}$ and $\mathbf{p}$ interchanged: see further below. That is to say, the 'photon' and monopole phase spaces are dual systems.

The stability subgroup G_0 giving rise to our coadjoint orbit $\mathcal{O}$ is a torus extension of the standard *little group* for massless particles E_2, so $\widetilde{\mathcal{P}}_0/H \simeq \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{S}^2$. However $\widetilde{\mathcal{P}}_0/E_2 \simeq \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{S}^3$, and this $\mathbb{S}^3$ sits over $\mathbb{S}^2$ like in the Hopf fibration.

3 Quaternionic Weyl systems

Weyl systems on a complex Hilbert space $\mathcal{H}$ are usually presented starting with a (real) symplectic vector space, say (V, ω), and a strongly continuous map $V \to \mathfrak{U}(\mathcal{H})$ to the unitary group on it, required to satisfy

$$W(v_1)W(v_2)W^\dagger(v_1)W^\dagger(v_2) = e^{i\omega(v_1,v_2)}\mathbb{I}. \tag{3.1}$$

Strong continuity, by means of the Stone-von Neumann theorem, implies that there exists a selfadjoint operator $R(v)$ such that

$$W(v) = e^{iR(v)}. \tag{3.2}$$

From the commutator relation we have

$$R\big([v_1, v_2]\big) - [R(v_1), R(v_2)] = i\omega(v_1, v_2)\mathbb{I}. \tag{3.3}$$

Another theorem by von Neumann says that Weyl systems exist for any finite dimensional symplectic vector space. They can be defined on a linear space of square integrable functions on any Lagrangian subspace of ω in V. If we denote by L such a Lagrangian subspace, we may write

$$(V, \omega) \rightleftharpoons (L \oplus L^*, \omega_0),$$

that is to say (V, ω) is symplectically isomorphic with $(T^*L, d\theta_0 \equiv \omega_0)$, where θ_0 is the Liouville 1-form on T^*L. By denoting $v \in V$ as $(y, \alpha) \in T^*L$, we have a Weyl system realized by

$$\begin{aligned} [W(0,\alpha)\psi](x) &= e^{i\langle\alpha,x\rangle}\psi(x); \\ [W(y,0)\psi](x) &= \psi(x+y). \end{aligned} \tag{3.4}$$

On defining either

$$W(y,\alpha) = W(y,0)W(0,\alpha) \qquad \text{or as} \qquad W(y,\alpha) = W(0,\alpha)W(y,0), \tag{3.5}$$

we find an *ordering* phase factor ambiguity. At the infinitesimal level we have a realization in terms of differential operators

$$iR(y,0) = \frac{\partial}{\partial x}; \qquad iR(0,\alpha) = \hat{x}. \tag{3.6}$$

The symplectic structure evaluated on vectors $(y,0)$ and $(0,\alpha)$ amounts to the commutator of the differential operators $\partial/\partial x$ and $\hat{x}$. In general, even though differential operators are unbounded, one prefers to see the algebra of operators acting on $\mathcal{H}$ realized as the algebra of differential operators acting on functions defined on L. In this framework it is more convenient to deal with square integrable functions considered as sections of an associated $U(1)$-bundle P over L. It is possible to write a function f on L as a function on P by setting $\tilde{f}(x,t) = f(x)e^{it}$. With this choice functions on L are associated with functions on P satisfying the equation

$$-i\frac{\partial}{\partial t}\tilde{f} = \tilde{f}; \tag{3.7}$$

then our algebra of differential operators may be realized in terms of vector fields, to wit $\partial/\partial x, -ix\partial/\partial t, -i\,\partial/\partial t$. These vector fields close the Lie algebra of the Heisenberg–Weyl group, with$-i\,\partial/\partial t$ generating the linear space of central elements. While sections of a line bundle are appropriate to describe spinless particles, to describe particles with an inner structure it is necessary to consider sections of some Hermitian complex vector bundle. Usually we consider $f : L \to \mathbb{C}^{2s+1}$ as functions associated with a trivialization of the $U(2s+1)$ Hermitian bundle P over L. In this setting the generators of our Weyl systems will be *matrix valued* differential operators.

The approach to quaternionic Quantum Mechanics undertaken more than forty years ago, may be considered from this perspective. Let $\mathbb{H}$ denote the field of quaternionic numbers

$$\mathbb{H} = \left\{ q = \sum_{\mu=0}^{3} q^\mu e_\mu \mid q^\mu \in \mathbb{R} \right\},$$

with their ordinary multiplication and involution. We will use the notations $1 = e_0$ and $\mathbf{e} = (e_1, e_2, e_3)$, so that $\mathbf{x}\cdot\mathbf{e} = \sum_{i=1}^{3} x^i e_i$ for $\mathbf{x} \in \mathbb{R}^3$. Note that $q^* q = \|q\|^2_{\mathbb{H}}$ defines the quaternion norm, and that $(\mathbf{x}\cdot\mathbf{e})^*(\mathbf{x}\cdot\mathbf{e}) = \|\mathbf{x}\|^2$. Then $\mathcal{H} \equiv \mathcal{L}^2(\mathbb{R}^3, d^3\mathbf{x}; \mathbb{H})$ is a Hilbert space of quaternion-valued functions. We consider wave functions realized on a Hilbert space (module) of quaternionic valued functions. By using the representation of quaternions by means of 2×2 skew-Hermitian matrices, we may write (when convenient)

$$e_0 = \sigma_0, \quad e_i = -i\sigma_i, \qquad \text{and} \qquad F(x) = f^0(x)e_0 + f^i(x)e_i. \tag{3.8}$$

The group $SU(2)$ acts on these fibres by conjugation and the vector bundle may be considered as an associated bundle with structure group $SU(2)$. If we identify $L \equiv \mathbb{R}^3$,

we may repeat our construction and lift vector fields from $\mathbb{R}^3$ to the total space of our vector bundle. To this aim we have to introduce a connection, that is, a procedure to lift vector fields to horizontal vector fields. We use the *gauge potential*

$$A = k\,\frac{[\mathbf{e}\cdot\mathbf{x}, \mathbf{e}\cdot d\mathbf{x}]}{\|\mathbf{x}\|^2}. \tag{3.9}$$

The origin of this choice may be traced back to the Hopf fibration $\pi : SU(2) \longrightarrow \mathbb{S}^2$: if we consider $s \in SU(2)$ we may define [11]:

$$\pi(s) = s^{-1}\sigma_3 s = \mathbf{x}\cdot\boldsymbol{\sigma} \tag{3.10}$$

and

$$\begin{aligned}\boldsymbol{\sigma}\cdot d\mathbf{x} &= -s^{-1}ds s^{-1}\sigma_3 s + s^{-1}\sigma_3 ds\\ &= -s^{-1}ds(\mathbf{x}\cdot\boldsymbol{\sigma}) + (\mathbf{x}\cdot\boldsymbol{\sigma})s^{-1}ds\\ &= [s^{-1}ds, \mathbf{x}\cdot\boldsymbol{\sigma}].\end{aligned}$$

Moreover, since $\mathbb{S}^2 \times \mathbb{R}_+ = \mathbb{R}^3 - \{0\}$, we may define a lifting which would consider *wave functions* as fields transforming covariantly under the rotation group, whose action in the inner space is by means of $SU(2)$. Given any $\mathbf{u} \in \mathbb{S}^2$, the translation $\mathbf{u}\cdot\partial/\partial\mathbf{x}$ is lifted to

$$\nabla_{\mathbf{u}} = e_0\mathbf{u}\cdot\frac{\partial}{\partial\mathbf{x}} + \frac{1}{2}\frac{[\mathbf{e}\cdot\mathbf{x}, \mathbf{e}\cdot\mathbf{u}]}{\|\mathbf{x}\|^2}, \tag{3.11}$$

considered as quaternionic-valued differential operator. Clearly

$$\nabla_{\mathbf{u}_1}\nabla_{\mathbf{u}_2} - \nabla_{\mathbf{u}_2}\nabla_{\mathbf{u}_1} = \Omega(\mathbf{u}_1, \mathbf{u}_2) \tag{3.12}$$

because

$$\left[\mathbf{u}_1\cdot\frac{\partial}{\partial\mathbf{x}}, \mathbf{u}_2\cdot\frac{\partial}{\partial\mathbf{x}}\right] = 0. \tag{3.13}$$

The curvature Ω is quaternion-valued and we may define the three presymplectic forms

$$e_1\Omega = \omega_1, \qquad e_2\Omega = \omega_2, \qquad e_3\Omega = \omega_3, \tag{3.14}$$

giving us the gCCR.

4 The Emch–Jadczyk calculus

The EJ model is an appropriate quantum framework for orbit-isospin coupling. In order to endow $\mathcal{H}$ with a complex linear structure we introduce: for every $\mathbf{x} \neq 0$, let $j(\mathbf{x})$ be the imaginary unit quaternion

$$j(\mathbf{x}) = \frac{\mathbf{e}\cdot\mathbf{x}}{\|\mathbf{x}\|}. \tag{4.1}$$

Then the linear operator J given by $(J\psi)(\mathbf{x}) = j(\mathbf{x})\psi(\mathbf{x})$ satisfies the relations $J^*J = I = JJ^*$ and $J^* = -J$, that is, it is unitary and skew–Hermitian; clearly, we also have $J^2 = -I$. Remark that the choice (4.1) defines a J invariant under rotations that commutes with the position operators; this is an easy calculation using $L_i = \epsilon_{ijk}x_j\partial_k - \frac{1}{2}e_i$ for the generator of rotations.

4.1 Noncommutative translations

On $\mathcal{H}$ one usually defines the translation by $\mathbf{a}$ by the operator $V(\mathbf{a})$ such that

$$[V(\mathbf{a})\psi](\mathbf{x}) = \psi(\mathbf{x} - \mathbf{a}),$$

but taking into account the character of rays rather than vectors of states one can also admit a *phase* factor. Here such translation is realized by the operator $U(\mathbf{a})$ defined by:

$$[U(\mathbf{a})\psi](\mathbf{x}) = w(\mathbf{a}; \mathbf{x} - \mathbf{a})\psi(\mathbf{x} - \mathbf{a}), \tag{4.2}$$

with $\mathbf{a} \in \mathbb{R}^3$. Here, for every $\mathbf{a}$, $w(\mathbf{a}; \mathbf{x})$ is the quaternion

$$w(\mathbf{a}; \mathbf{x}) = \cos(\alpha/2) + j(\mathbf{x} \times \mathbf{a}) \sin(\alpha/2) = \exp\left[j(\mathbf{x} \times \mathbf{a})\alpha/2\right], \tag{4.3}$$

with α being the angle between $\mathbf{x}$ and $\mathbf{x} + \mathbf{a}$. If we use w to define the linear operator $W(\mathbf{a})$ by

$$[W(\mathbf{a})\psi](\mathbf{x}) = w(\mathbf{a}; \mathbf{x})\psi(\mathbf{x}), \quad \text{a.e.} \tag{4.4}$$

and then

$$U(\mathbf{a}) = V(\mathbf{a})W(\mathbf{a}). \tag{4.5}$$

Some properties of w which will be useful below are:

- $w(0; \mathbf{x}) = 1$.
- $w(\mathbf{a}; \mathbf{x})w^*(\mathbf{a}; \mathbf{x}) = 1$.
- $w(\mathbf{a}; \mathbf{x} - \mathbf{a}) = w^*(-\mathbf{a}; \mathbf{x})$.
- $w(t\mathbf{a}; \mathbf{x} + s\mathbf{a})w(s\mathbf{a}; \mathbf{x}) = w((s + t)\mathbf{a}; \mathbf{x})$.

Let now $\mathbf{u} \in \mathbb{S}^2$. We can define generators for the continuous unitary representation $U(s\mathbf{u})$ by

$$\nabla_{\mathbf{u}} = \lim_{t \downarrow 0} \left[\frac{d}{dt} U(t\mathbf{u})\psi(\mathbf{x})\right] = \lim_{t \downarrow 0} \left[\frac{d}{dt} w(t\mathbf{u}; \mathbf{x} - t\mathbf{u})\psi(\mathbf{x} - t\mathbf{u})\right]. \tag{4.6}$$

One obtains

$$\nabla_{\mathbf{u}} = \left(\mathbf{u} \cdot \boldsymbol{\partial} + \frac{1}{2}\mathbf{e} \cdot \frac{\mathbf{u} \times \mathbf{x}}{||\mathbf{x}||^2}\right),$$

whereupon we recognize the sum of the (non-commuting) infinitesimal generators of V and W, respectively. Thus, with the obvious meaning for the X_i, we readily compute the following commutation relations

$$\begin{aligned} [\nabla_i, X_j] &= \delta_{ij}, \\ [\nabla_i, \nabla_j] &= -\tfrac{1}{2} J\, \epsilon_{ijk} \frac{x^k}{\|\mathbf{x}\|^3}, \\ [X_i, X_j] &= 0, \end{aligned}$$

that should be compared with (1.1) and (2.2).

The key result of the EJ calculus is that the so defined operators $U(\mathbf{a})$ actually define a locally operating projective representation of the translation group [10], i.e.

$$U(\mathbf{a})U(\mathbf{b}) = U(\mathbf{a}+\mathbf{b})M(\mathbf{a},\mathbf{b}).$$

Here $M(\mathbf{a},\mathbf{b})$ is a phase factor multiplication

$$[M(\mathbf{a},\mathbf{b})\psi](\mathbf{x}) = m(\mathbf{a},\mathbf{b},\mathbf{x})\,\psi(\mathbf{x})$$

with $m(\mathbf{a},\mathbf{b},\mathbf{x})$ being given by

$$m(\mathbf{a},\mathbf{b},\mathbf{x}) = \exp\big(JS(\mathbf{a},\mathbf{b},\mathbf{x})\big),$$

where S denotes the (product of the monopole strength) by the area of the triangle spanned by $\mathbf{x}, \mathbf{x}+\mathbf{a}, \mathbf{x}+\mathbf{a}+\mathbf{b}$. This guarantees associativity: $U(\mathbf{a})\big[U(\mathbf{b})U(\mathbf{c})\big] = [U(\mathbf{a})U(\mathbf{b})]U(\mathbf{c})$ (see next section).

Except for the presence of the quaternionic complex structure J, this is similar to the Moyal product, which bodes well for the quantization/dequantization procedure.

5 Exponential representation of the Weyl system

In the usual case Weyl systems are represented as exponential as in (3.2). It is useful to express also the quaternionic Weyl system as an exponential. A first problem is that in the quaternionic context there is no single imaginary unit. Nevertheless EJ have provided the solution of the problem in the function $j(\mathbf{x})$ defined in (4.1), and its operatorial counterpart J. We can therefore define the operator

$$P_i = J\nabla_i = -J\left(\partial_i + \frac{1}{2}\frac{\epsilon_{ijk}x_j e_k}{\|\mathbf{x}\|^2}\right) \tag{5.1}$$

it is possible to prove that J commutes with ∇_i and therefore $[P_i, P_j] = (1/2)\epsilon_{ijk}(x^k/\|\mathbf{x}\|^3)$. Therefore P_i is the generator of translations in the quaternionic Hilbert space. Notice that the two summands in P_i do not commute.

Finally, let us consider the product of two finite translations

$$(U(\mathbf{a})U(\mathbf{b})\psi)(\mathbf{x}) = (U(\mathbf{a})\psi')(\mathbf{x}) = w(\mathbf{a};\mathbf{x}-\mathbf{a})\psi'(\mathbf{x}-\mathbf{a}) = w(\mathbf{a};\mathbf{x}-\mathbf{a})w(\mathbf{b};\mathbf{x}-\mathbf{a}-\mathbf{b})\psi(\mathbf{x}-\mathbf{a}-\mathbf{b}).$$

On the other hand,

$$(U(\mathbf{a}+\mathbf{b})M(\mathbf{a},\mathbf{b})\psi)(\mathbf{x}) = w(\mathbf{a}+\mathbf{b};\mathbf{x}-\mathbf{a}-\mathbf{b})(M(\mathbf{a},\mathbf{b})\psi)(\mathbf{x}-\mathbf{a}-\mathbf{b}), \tag{5.2}$$

with $M(\mathbf{a},\mathbf{b})$ defined as

$$(M(\mathbf{a},\mathbf{b})\psi)(\mathbf{x}) = m(\mathbf{a},\mathbf{b};\mathbf{x})\psi(\mathbf{x}) = w^*(\mathbf{a}+\mathbf{b},\mathbf{x})w(\mathbf{a};\mathbf{x}+\mathbf{b})w(\mathbf{b};\mathbf{x})\,\psi(\mathbf{x}). \tag{5.3}$$

Since $w(\mathbf{a};\mathbf{x}) = 1$ and $w(0;\mathbf{x}-a) = w^*(\mathbf{a};\mathbf{x})$ we have that $m(\mathbf{a},\mathbf{b};\mathbf{x})$ satisfies

$$m(\mathbf{a},-\mathbf{a};\mathbf{x}) = 1. \tag{5.4}$$

We obtain then

$$U(\mathbf{a})U(\mathbf{b}) = U(\mathbf{a}+\mathbf{b})M(\mathbf{a},\mathbf{b}). \tag{5.5}$$

We now construct a Weyl system from $\mathbf{P}$ and $\mathbf{X}$. Consider the operator

$$T(\alpha) = e^{J[\mathbf{a}\cdot\mathbf{P}+\mathbf{a}'\cdot\mathbf{X}]} = e^{J\mathbf{a}\cdot\mathbf{P}}e^{J\mathbf{a}'\cdot\mathbf{X}}e^{\frac{1}{2}a^i a'^j[P^i,X^j]} = e^{J\mathbf{a}\cdot\mathbf{P}}e^{J\mathbf{a}'\cdot\mathbf{X}}e^{-\frac{1}{2}J\mathbf{a}\cdot\mathbf{a}'} = e^{J\mathbf{a}'\cdot\mathbf{X}}e^{J\mathbf{a}\cdot\mathbf{P}}e^{\frac{1}{2}J\mathbf{a}\cdot\mathbf{a}'} \tag{5.6}$$

with $\alpha = (\mathbf{a},\mathbf{a}')$. Remember that $\exp(J\mathbf{a}\cdot\mathbf{P}) \equiv U(\mathbf{a})$. We have then

$$(T(\alpha)\psi)(\mathbf{x}) = (e^{J\mathbf{a}\cdot\mathbf{P}}e^{J\mathbf{a}'\cdot\mathbf{X}}e^{-\frac{1}{2}J\mathbf{a}\cdot\mathbf{a}'}\psi)(\mathbf{x}) = w(\mathbf{a};\mathbf{x}-\mathbf{a})e^{j(\mathbf{x}-\mathbf{a})\mathbf{a}'\cdot(\mathbf{x}-\mathbf{a})}e^{-\frac{1}{2}j(\mathbf{x}-\mathbf{a})\mathbf{a}\cdot\mathbf{a}'}\psi(\mathbf{x}-\mathbf{a}), \tag{5.7}$$

but also

$$(T(\alpha)\psi)(\mathbf{x}) = (e^{J\mathbf{a}'\cdot\mathbf{X}}e^{J\mathbf{a}\cdot\mathbf{P}}e^{\frac{1}{2}J\mathbf{a}\cdot\mathbf{a}'}\psi)(\mathbf{x}) = e^{j(\mathbf{x})\mathbf{a}'\mathbf{x}}w(\mathbf{a};\mathbf{x}-\mathbf{a})e^{\frac{1}{2}j(\mathbf{x}-\mathbf{a})\mathbf{a}\cdot\mathbf{a}'}\psi(\mathbf{x}-\mathbf{a})\,. \tag{5.8}$$

We compute

$$T(\alpha)T(\beta) = e^{J[\mathbf{a}\cdot\mathbf{P}+\mathbf{a}'\cdot\mathbf{X}]}e^{J[\mathbf{b}\cdot\mathbf{P}+\mathbf{b}'\cdot\mathbf{X}]} = e^{J\mathbf{a}\cdot\mathbf{P}}e^{J\mathbf{a}'\cdot\mathbf{X}}e^{J\mathbf{b}\cdot\mathbf{P}}e^{J\mathbf{b}'\cdot\mathbf{X}}e^{-\frac{1}{2}J(\mathbf{a}\cdot\mathbf{a}'+\mathbf{b}\cdot\mathbf{b}')}\,. \tag{5.9}$$

On using (5.5) and (5.6) we arrive at the Weyl system

$$T(\alpha)T(\beta) = T(\alpha+\beta)M(\mathbf{a},\mathbf{b})\exp\Big(\frac{1}{2}J(\mathbf{a}\cdot\mathbf{b}'-\mathbf{b}\cdot\mathbf{a}')\Big)\,. \tag{5.10}$$

We see that this Weyl system provides as usual, but there are two phases. The first is the antisymmetric product of the two vectors in $\mathbb{R}^6$. This would be present also in the absence of the monopole, it gives the noncommutativity of position and momenta, however with the ‘imaginary’ unit given by the quaternionic radial functions $j(\mathbf{x})$. The factor M instead is the one which contains the information on the noncommutativity of the translations. Both phases are of course crucial for the description of the particle in the field of a monopole.

6 Outlook

Quaternions are well suited to describe the quantization of classical particles in the presence of the field of a magnetic monopole. We have laid the basis for this quantization. It is in principle possible (and will be presented elsewhere) to construct a full deformation quantization. One can built a full Weyl map which associates functions on phase space with operators on the quaternionic Hilbert space. The quantized functions belong to a subalgebra, in such a way that the map is (at least formally) invertible, and therefore

provides a star product. Possible colour-breaking phenomena —see [12, 13]— are to be fit in the formalism; indeed, there are plenty of questions here we should have answered long ago. But "the gaps in the knowledge of the wise has been filled even so slowly" [14].

Note Added After this work had appeared we constructed the star product quantizing the motion of particle in a monopole field in [15].

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